Group Theory 15

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Theorem:- If G is a finite p-group and r_s is the number of subgroups of G having order p^s then $r_s \equiv 1 \pmod p$

Lemma:- (Landau):-

Given $n > 0$ and $q \in \mathbb{Q}$ there are only finitely many n-tuples $(i_1, i_2, \ldots, i_n)$ of positive integers such that
$$q = \sum_{j=1}^{n} (1/i_j)$$

Proof:- By Induction

Theorem:- For every $n \geq 1$, there are only finitely many finite groups having exactly n conjugacy classes

Proof:- G be the group with n-conjugacy classes and $|Z(G)| = m$
$$|G| = |Z(G)| + \sum_{j=m+1}^{n} [G : C_G(x_j)]$$

If $i_j = |G|$ for $1 \leq j \leq m$ and $i_j = |G| / [G : C_G(x_j)] = |C_G(x_j)|$ for $m+1 \leq j \leq n$ then $1 = \sum (1/i_j)$

$\Rightarrow \exists$ only finitely many such tuples (by previous lemma)

Sylow:-

Definition:- If p is a prime then a Sylow p-subgroup P of a group G is a maximal p-subgroup.

Let P be a Sylow p-subgroup of a finite group

Lemma :- Let P be a Sylow p-subgroup of a finite group G

(i) $|N_G(P)/P|$ is prime to p.

(ii) If $a \in G$ has order some power of p and $aPa^{-1} = P$ then $a \in P$

Proof :- (i) Suppose $p \,\big|\, |N_G(P)/P|$ then $N_G(P)/P$ will have some element Pa of order $p \Rightarrow \langle Pa \rangle$ has order p

$S \leq N_G(P) \leq G$ that contains P. $S/P \cong \langle Pa \rangle$

S and $\langle Pa \rangle$ are both p-groups $\Rightarrow$ that P is not maximal. $\Rightarrow \Leftarrow$

(ii) $\mathrm{ord}(a) = p^m$ $\quad aPa^{-1} = P$

$a \in N_G(P)$. If $a \notin P$ then $Pa \in N_G(P)/P$ and Pa as order $p \Rightarrow \Leftarrow$

Theorem (Sylow) :-

(i) If P is a Sylow p-subgroup of a finite group G, then all Sylow p-subgroups of G are conjugate to P

(ii) If there are r-Sylow p-subgroups, then r is a divisor of $|G|$ and $r \equiv 1 \mod p$

Proof :- $C = \{P_1, \ldots, P_k\}$ be the set of conjugates of P and $P_1 = P$

$\phi : G \to S_C \qquad \phi_a(P_i) = aP_ia^{-1}$

$a \mapsto \phi_a$

Let Q be a Sylow p-subgroup of G. ϕ acts only on Q then we get that, every orbit of C under this action has order dividing $|Q| \Rightarrow$ order of orbit of C is of form p^m

$\exists P_i$ such that $\phi_a(P_i) = P_i \ \forall a \in Q$.
$\Rightarrow aP_ia^{-1} = P_i \ \forall a \in Q$
If $a \in Q$ then $a \in P_i \Rightarrow Q \leq P_i$
So as Q is Sylow $Q = P_i$.

If $Q = P$ then $|C| = r \equiv 1 \bmod p$
If Q is not a conjugate of P, i.e., $Q \notin C$
then $\{P_i\}$ has a Q-orbit of size 1 and $Q = P_i$ for some $i \Rightarrow \Leftarrow$ as $Q \neq P_i$ here.
$p \mid |C|$ as it has to be p^m form
$r \equiv 0 \bmod p \Rightarrow \Leftarrow$
So we get that, every Sylow p-subgroup Q is a conjugate of P

- A finite group G has a unique Sylow p-subgroup P for some prime p iff $P \triangleleft G$.

<u>Theorem</u>:- If G is a finite group of order $p^k n$ where $\gcd(p, n) = 1$ then every Sylow p-subgroup P of G has order p^k

<u>Proof</u>:- $[G:P] = [G:N_G(P)][N_G(P):P]$.
Now, $[G:N_G(P)] = r$ = no of conjugates of P
So, $\equiv 1 \bmod p$ so $\gcd([G:N_G(P)], p) = 1$
$[N_G(P):P] = |N_G(P)/P|$ is also coprime with p
So we get $|P| = p^m$, $m \leq k$
$\Rightarrow [G:P] = |G|/|P| = p^{k-m} n$
Now $[G:P]$ is coprime with $p \Rightarrow k-m = 0$

So $[G : P] = n$

$\Rightarrow$ Every Sylow p-subgroup of G has order p^k.

•> Let G be a finite group and let p be a prime. If $p^k \mid |G|$ then G contains a subgroup of order p^k

Theorem :- (Frattini) :-
Let K be a normal subgroup of a finite group G. If P is a Sylow p-subgroup of K for some prime p then $G = K N_G(P)$

Q> Let H be a Sylow p-subgroup of a finite group G. We have $N_G(H) = \{x \in G : x^{-1}Hx = H\}$. Prove that H is the only Sylow p-subgroup of G contained in $N_G(H)$.

Ans :— $h \in H$, $h^{-1}Hh = H$, $h \in N_G(H)$

$H \subset N_G(H)$, $H \triangleleft N_G(H)$ also

H is a normal Sylow p-subgroup of $N_G(H)$.

$\Rightarrow H$ is the only Sylow p-subgroup of G in $N_G(H)$.

Q> Let H be a Sylow p-subgroup of a finite group G. Let $x \in N_G(H)$ such that $Ord(x) = p^n$ for some positive integer n. Prove that $x \in H$.

Ans :— $x \in N_G(H)$ and $Ord(x) = p^n$, $\langle x \rangle \subseteq N_G(H)$ $\Rightarrow$ Sylow p-subgroup

So $x \in H$ as H is the only Sylow p-subgroup

Q> Let G be a group of order p^2 then prove that

Q> Let G be a group of order p^2 then prove that G is abelian

Ans:- $Ord(G) = p^2 \Rightarrow Ord(Z(G)) = 1, p, p^2$ (1 → done)

$Ord(Z(G)) = p \Rightarrow Ord(G/Z(G)) = p \Rightarrow G/Z(G)$ is cyclic
$\Rightarrow G$ is abelian

$Ord(Z(G)) = p^2 \Rightarrow G$ is abelian

Q> Let G be a non-Abelian group of order $2^2 3^2 = 36$. Prove that G has more than one Sylow 2-subgroup or more than one Sylow 3-subgroup

Ans:- Suppose G denies the argument.
So G will have exactly one Sylow 3-subgroup say H and exactly one Sylow 2-subgroup say K.

$H \triangleleft G$ and $K \triangleleft G$

$Ord(H) = 3^2$ and $Ord(K) = 2^2$

$gcd(2^2, 3^2) = 1$ so we get $H \cap K = \{e\}$

$Ord(HK) = 2^2 3^2 = Ord(G)$

$\Rightarrow HK = G$

Now H and K are abelian $\Rightarrow G$ is also abelian as $G \cong H \oplus K$

$\Rightarrow \Leftarrow$

So the argument is true

Q> Let H be a subgroup of a group G. Prove that H is normal in G iff $g^{-1}Hg \subset H$ for each $g \in G$.

Ans:- H is normal
$\Rightarrow gH = Hg \quad \forall g \in G$

$a \in gHg^{-1} \Rightarrow a = gh_1g^{-1}$ for some $h_1 \in H$

$a = g h_1 g^{-1} = h_1 g g^{-1} = h_1 \in H$

So $g^{-1} H g \subset H$

Converse, $g^{-1} H g \subset H$ for each $g \in G$

We need to show $H \subset g^{-1} H g$ for each $g \in G$

$h \in H$ and $g \in G$, $g H g^{-1} \subset H \Rightarrow g h g^{-1} \in H$

$g^{-1} (g h g^{-1}) g = h \in g^{-1} H g$

$\Rightarrow H \subset g^{-1} H g$ for each $g \in G$

So $g H g^{-1} = H \Rightarrow H$ is normal